

**XY model in small-world networks**Beom Jun Kim,<sup>1</sup> H. Hong,<sup>2</sup> Petter Holme,<sup>1</sup> Gun Sang Jeon,<sup>3</sup> Petter Minnhagen,<sup>1</sup> and M. Y. Choi<sup>2</sup><sup>1</sup>*Department of Theoretical Physics, Umeå University, 901 87 Umeå, Sweden*<sup>2</sup>*Department of Physics, Seoul National University, Seoul 151-747, Korea*<sup>3</sup>*Center for Strongly Correlated Materials Research, Seoul National University, Seoul 151-747, Korea*

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The phase transition in the XY model on one-dimensional small-world networks is investigated by means of Monte Carlo simulations. It is found that long-range order is present at finite temperatures, even for very small values of the rewiring probability, suggesting a finite-temperature transition for any nonzero rewiring probability. Nature of the phase transition is discussed in comparison with the globally coupled XY model.

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Examples of complex networks are abundant in many disciplines of science and have recently attracted much attention [1]. Among interesting phenomena is the so-called *small-world effect* that a randomly chosen pair of nodes may be connected by a remarkably small number of intervening nodes. This effect, first noticed by Watts and Strogatz (WS) [2], may be observed in a variety of real network systems such as the world-wide web, social networks, and scientific collaboration networks [3,4]. The WS model is based on a locally highly connected regular network, in which some of the links are randomly “rewired,” creating long-range “shortcuts.” In such a model network, the small-world effect is usually measured by the scaling behavior of the characteristic path length  $\ell$ , defined to be the average of the shortest distance between two nodes:  $\ell \sim \log N$  with the network size (i.e., the number of nodes)  $N$ . Noteworthy, the small-world phenomena in the WS model emerge even at very small values of the rewiring probability  $P \sim \mathcal{O}(N^{-1})$ , which implies that the global feature of the network is altered dramatically in the presence of only a tiny fraction of shortcuts.

From the point of view of statistical physics, the above small-world effect may imply the emergence of global coherence in the presence of shortcuts: In the absence of shortcuts, global coherence is difficult to achieve since the information to make each element have the same state should travel long distance of the order of the network size. As the number of shortcuts is increased, on the other hand, long-range connections become available, assisting the system to behave as a whole. The significance of such a shorter global length scale has been tested with many statistical physical problems studied on a small-world network topology. Included are the signal propagation speed [2], synchronizability [2], dynamics of Hodgkin-Huxley neurons [5], epidemiology [6,7], percolation [7–9], and—most relevant to the present paper—the Ising model, where a nonvanishing order parameter has been demonstrated in the presence of a vanishingly small fraction of shortcuts [10].

The XY model, describing two-dimensional (2D) spin (i.e., planar rotor) systems or superconductors and superfluids, is one of the most well-known systems in statistical physics. In one and two spatial dimensions, the Mermin-Wagner theorem predicts that the XY model with only local

interactions should not possess long-range order at any nonzero temperatures. Nevertheless, in two dimensions, it has been shown that although true long-range order is not possible, quasi-long-range order appears at finite temperatures [11]. As the XY model includes more and more long-range interactions, one expects that the system may display true long-range order. In this paper, we study the XY model on the WS model of small-world networks and investigate the possibility of long-range order as the number of shortcuts or long-range interactions is increased.

The Hamiltonian for the XY model reads

$$H = -\frac{1}{2} \sum_{i \neq j} J_{ij} \cos(\phi_i - \phi_j), \quad (1)$$

where  $\phi_i$  is the angular direction of the 2D spin or the phase of the superconducting order parameter at node  $i$ . The coupling matrix  $J_{ij}$  is given by

$$J_{ij} = J_{ji} \equiv \begin{cases} J, & \text{if } i \text{ and } j \text{ are connected,} \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

For example, in the standard 2D XY model on a square lattice, we have  $J_{ij} = 0$  except for the four nearest neighbors of site  $i$ . In the WS model for the small-world network [2], the regular network with  $N$  sites is first constructed by connecting each site to its  $2k$  nearest neighbors. Each local link is visited once, and then with the rewiring probability  $P$  it is removed and reconnected to a randomly chosen site. In this manner, a small-world network with size  $N$  is constructed for given  $k$  and  $P$ . We then study the phase transition of the XY model on this small-world network by measuring the order parameter

$$\langle m \rangle \equiv \left\langle \left| \frac{1}{N} \sum_j e^{i\phi_j} \right| \right\rangle, \quad (3)$$

where  $\langle \dots \rangle$  denotes the thermal average and the average over different network realizations, denoted by  $[\dots]$ , is also to be taken. When  $P = 0$ , the system behaves as the one-dimensional (1D) XY model, displaying no long-range order at finite temperatures. For  $P \neq 0$ , the system has many long-range interactions in the thermodynamic limit and the Mermin-Wagner theorem cannot disprove the existence of

long-range order. In particular, for  $P=1$ , it is on a random network with the number of randomly connected links being  $kN$ .

We perform extensive Monte Carlo (MC) simulations with the standard Metropolis local update algorithm, to measure various thermodynamic quantities and moments of the order parameter. For a thorough analysis of the MC results, we detect the phase transition in three different methods. First, Binder's fourth-order cumulant [12],

$$U_N(T) \equiv 1 - \frac{[\langle m^4 \rangle]}{3[\langle m^2 \rangle]^2} \quad (4)$$

for different sizes  $N$  should have a unique crossing point at the transition temperature  $T_c$ . Expanding the cumulant near  $T_c$ , we write

$$U_N(T) \approx U^* + U_1 \left( 1 - \frac{T}{T_c} \right) N^{1/\bar{\nu}} \quad (5)$$

or

$$\Delta U_N \equiv U_N(T_1) - U_N(T_2) \propto N^{1/\bar{\nu}}, \quad (6)$$

with  $T_1$  and  $T_2 (> T_1)$  picked near  $T_c$ . From the above expression, the critical exponent  $\bar{\nu}$ , describing the divergence of the correlation volume  $\xi_V$  at  $T_c$  [13,14]

$$\xi_V \sim |T - T_c|^{-\bar{\nu}}, \quad (7)$$

may be determined. Our second method is based on the finite-size scaling of the order parameter, which exhibits the critical behavior  $[\langle m \rangle] \sim (T_c - T)^\beta$  in the thermodynamic limit. In a finite-sized system, we expect the behavior to  $[\langle m \rangle] = (T_c - T)^\beta f(T, N)$  with a function  $f$  of at most two arguments  $T$  and  $N$ . We then use the standard finite-size scaling idea that the ratio of the correlation volume to the system size gives the argument of the scaling function, and obtain the finite-size scaling form

$$[\langle m \rangle] = N^{-\beta/\bar{\nu}} g((T - T_c)N^{1/\bar{\nu}}), \quad (8)$$

which leads to a unique crossing point at  $T_c$  in the plot of  $[\langle m \rangle]N^{\beta/\bar{\nu}}$  versus  $T$ . As discussed later in the present paper, the phase transition on the small-world network is of the mean-field nature, which provides us the third method of locating the transition temperature: With the mean-field value of the critical exponent  $\alpha=0$ , the specific heat neither diverges nor vanishes, remaining finite possibly with jump discontinuity. Accordingly, it is plausible to write the finite-size scaling form of the specific heat as

$$C_v = h((T - T_c)N^{1/\bar{\nu}}), \quad (9)$$

which again crosses at  $T_c$ . In the same way as in Eq. (5), the expansion of  $C_v$  near  $T_c$  may be used to determine  $\bar{\nu}$ .

All the above finite-size scaling forms are based on the assumption that we have only two spatial scales in the system: the size  $N$  and the correlation volume  $\xi_V$  diverging at

$T_c$ . In fact, it is known that the small-world network has an additional length scale, the typical distance between the ends of shortcuts, given by  $\zeta = (kP)^{-1}$  [8]. Accordingly, in the presence of the three competing scales  $(N, \xi, \zeta)$ , the standard finite-size scaling function should take the form  $\chi(\xi/N, \zeta/N)$  [8,15]. Here, we focus on sufficiently large systems with  $N$  much larger than  $\zeta$ , where  $\chi(\xi/N, \zeta/N)$  may be approximated as  $\chi(\xi/N, 0)$ . This leads to the above-mentioned finite-size scaling forms without  $\zeta$  [Eqs. (5), (8), and (9)], which become more precise as the system size grows. This is manifested below when the MC results are presented.

The small-world network, on which the  $XY$  model is defined, is characterized by the rewiring probability  $P$  and the local interaction range  $k$ . We focus only on the case  $k=3$ ; however, we believe that qualitatively, the same conclusion should hold for other values of  $k (> 1)$  [16]. From the MC simulations, we have confirmed that the network with  $P=0$  does not exhibit long-range order at finite temperatures, in accordance with the Mermin-Wagner theorem. In the opposite case with  $P=1$ , where all  $kN (= 3N)$  connections are long-ranged shortcuts, the system is found to undergo a well-defined finite-temperature transition. We first present our MC results for  $P=0.2$  in Fig. 1. The finite-size scaling of Binder's cumulant, the order parameter, and the specific heat all reveals unanimously  $T_c \approx 2.235$  in units of  $J/k_B$ , confirming the presence of a finite-temperature phase transition in the  $XY$  model on the small-world network. In particular, the obtained critical exponents  $\beta=1/2$ ,  $\bar{\nu}=2$ , and  $\alpha=0$  establish the mean-field nature of the transition. Also confirmed is the validity of our finite-size scaling forms, with the additional length scale  $\zeta$  reflected in the drift of the crossing points at small sizes. As the size  $N$  is increased, the finite-size effects associated with  $\zeta$  reduce and the finite-size scaling forms based on the assumption of only two scales  $N$  and  $\xi_V$  prevail, leading to well-defined crossing points at larger sizes.

For comparison, we also study the globally coupled  $XY$  model, where  $J_{ij} = J = O(1/N)$  for all  $i$  and  $j$  and  $T$  is measured in units of  $NJ/k_B$ . We use the saddle-point method accompanied by the Hubbard-Stratonovich transformation [17], and obtain various thermodynamic quantities such as the specific heat and the order parameter in Eq. (3). We also perform the MC simulation and compare the results in Fig. 2, which exhibits excellent correspondence between the two. As expected, the globally-coupled  $XY$  model indeed possesses the mean-field transition, which is the same as that in the  $XY$  model on the small-world network. It should be noted that the number of connected links (i.e., the number of nonzero elements in the interaction matrix  $J_{ij}$ ) in the globally coupled  $XY$  model is given by  $N(N-1)/2$ , which is in sharp contrast with the number  $O(N)$  in the small-world networks. This indicates that the  $XY$  model can be made to have global phase coherence (i.e., true long-range order) at finite temperatures by rewiring  $O(N)$  connections, with the total number of interactions still kept  $O(N)$  instead of  $O(N^2)$  as in the globally coupled  $XY$  model.

What is the minimum value of  $P$  to have such long-range order at finite temperatures? To answer this question, we

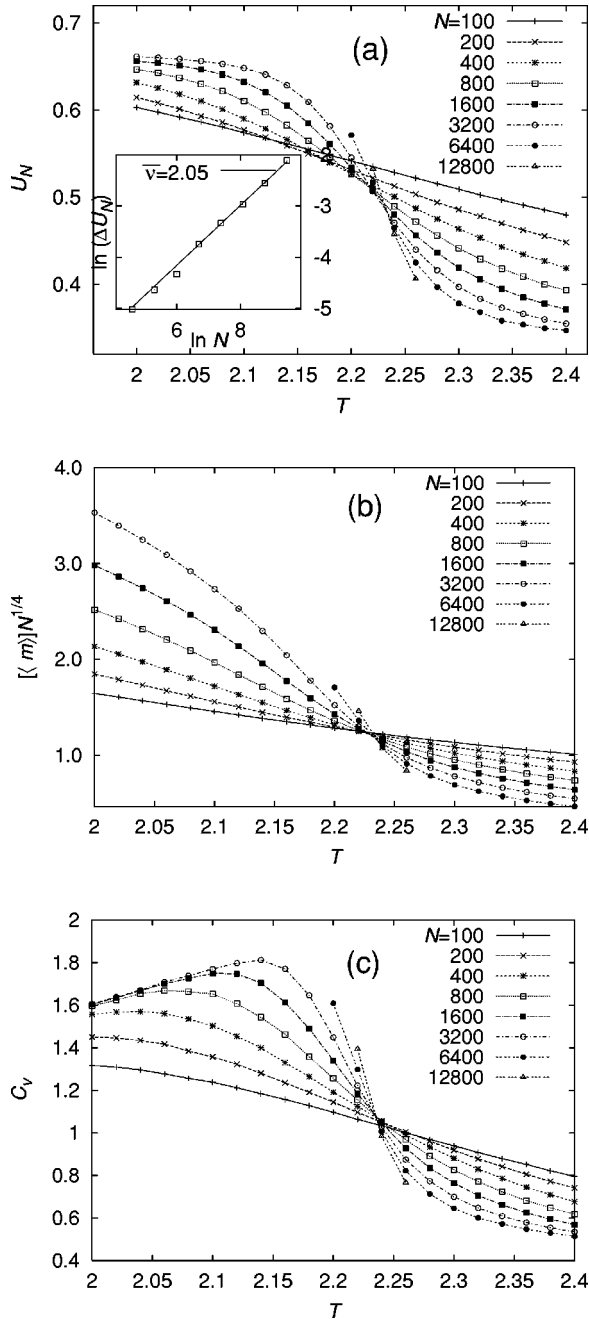


FIG. 1. The XY model on the small-world network with  $P = 0.2$  and  $k = 3$  displays a mean-field phase transition at the finite temperature  $T_c$ : (a) Binder's cumulant  $U_N$  has a unique crossing point at  $T_c \approx 2.235$  (in units of  $J/k_B$ ) and the critical exponent  $\bar{\nu} \approx 2.0$  is obtained from Eq. (6) (see the inset). (b) Finite-size scaling of the order parameter [see Eq. (8)] again yields  $\beta \approx 0.5$  and  $\bar{\nu} \approx 2.0$  with  $T_c \approx 2.235$ . (c) Specific heat  $C_v$  (in units of  $k_B$ ) also has a crossing point at  $T_c \approx 2.235$ , as implied by Eq. (9).

have also performed MC simulations at various values of  $P$ . As the rewiring probability  $P$  is reduced, the finite-size effects due to the length scale  $\zeta$  grow and become no more negligible, making it necessary to perform simulations on larger systems. Since it is observed that the finite-size effects are much more discernible in Binder's cumulant and in the

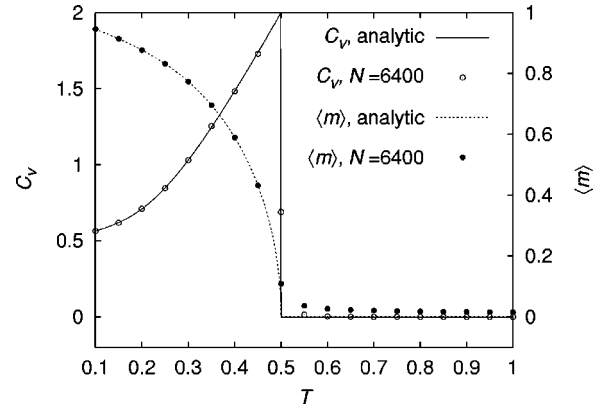


FIG. 2. Specific heat  $C_v$  (in units of  $k_B$ ) and the order parameter  $\langle m \rangle$  of the globally coupled XY model. Compared are  $C_v$  and  $\langle m \rangle$  from analytic calculation (full and dotted lines, respectively) and from Monte Carlo simulations for the system size  $N = 6400$  (empty squares and filled circles). The phase transition is of the mean-field type characterized by  $\langle m \rangle \sim (T_c - T)^\beta$  with  $\beta = 1/2$  and the jump in  $C_v$  at the transition ( $T_c = 1/2$  in units of  $JN/k_B$ ).

specific heat than in the order parameter, we estimate  $T_c$  only from the crossing point in the plot of  $[\langle m \rangle]N^{1/4}$  versus  $T$ . For  $P \geq 0.03$ , we have confirmed that as the system size grows, all three methods to determine  $T_c$  give the same value. As an example, we show in Fig. 3 the determination of  $T_c$  for  $P = 0.05$ .

Finally, Fig. 4 summarizes the results and presents the phase diagram for the XY model on small-world networks with  $k = 3$ . We observe that the phase boundary is very well described by the logarithmic form:  $T_c(P) \approx 0.41 \ln P + 2.89$ , as shown in the inset of Fig. 4, without an obvious reason. A naive extrapolation of this form predicts a very small value of the critical rewiring probability:  $P_c \approx 0.001$ . It is, however, likely that there takes place deviation from the logarithmic dependence at small  $P$  and any nonzero rewiring probability ( $P > 0$ ) presumably supports long-range order at sufficiently low but finite temperatures, resulting in  $P_c = 0$ . One may provide a simple argument in favor of this expectation. Since the spins separated within the correlation length are correlated with each other, the presence of shortcuts in such a way that the typical distance  $\zeta$  between the ends of shortcuts is smaller than the correlation length  $\xi$  (of the 1D system) does not affect the system substantially, leaving the 1D nature intact. When  $\zeta$  grows beyond the correlation length, on the other hand, the long-range interactions via shortcuts come into play, giving rise to the mean-field character. Accordingly, the crossover between the 1D behavior and the mean-field one is given by the condition  $\xi \approx \zeta$ , which also describes the so-called small-world transition between the two geometrical regimes, the large-world regime, and the small-world one [18]. Recognizing the low-temperature behavior of the correlation length  $\xi \sim T^{-1}$  [19] in one dimension ( $P = 0$ ) and the typical distance  $\zeta \sim P^{-1}$ , we thus expect the behavior of the transition temperature  $T_c \sim P$  in the limit of small  $P$  [20]. Here, it is of interest to note the difference from the Ising model on small-world networks. The latter exhibits the behavior  $T_c \sim -(\ln P)^{-1}$  [10], which originates from the

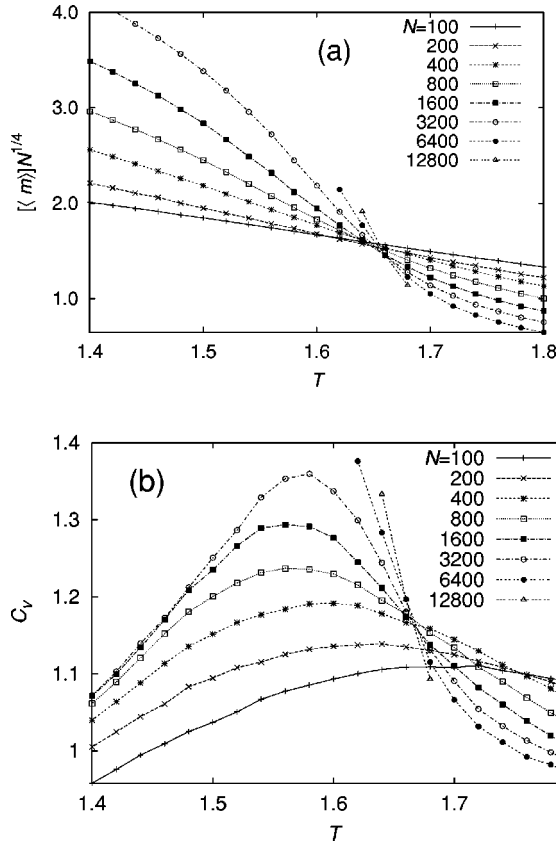


FIG. 3. XY model on the small-world network for  $P=0.05$ . From the crossing points in the scaling forms of (a) the order parameter  $[\langle m \rangle]$  and (b) the specific heat  $C_v$  (in units of  $k_B$ ),  $T_c \approx 1.66$  (in units of  $J/k_B$ ) is obtained. The drift of the crossing point at small sizes is larger in the specific heat (b) than in the order parameter (a).

exponential temperature dependence in the 1D Ising model,  $\xi \sim e^{a/T}$  with the constant  $a$  proportional to the coupling strength.

In conclusion, we have studied the XY model on small-world networks through the use of the standard Monte Carlo simulation method. At all nonzero values of the rewiring probability  $P$  considered in this paper, the finite-temperature transitions have been detected by Binder's cumulant, the specific heat, and the order parameter, together with appropriate finite-size scaling forms. The phase transition has been

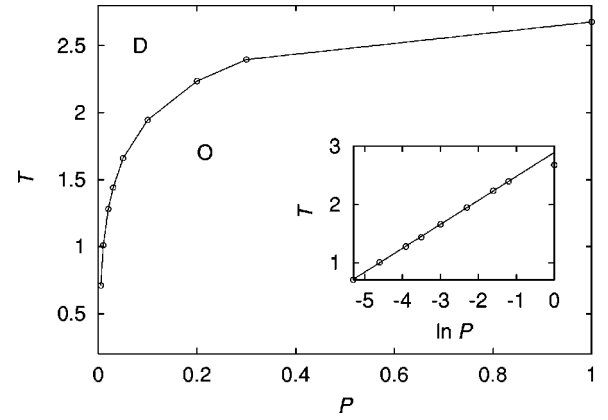


FIG. 4. The phase diagram of the XY model on the small-world network with  $k=3$ , drawn on the plane of the rewiring probability  $P$  and the temperature  $T$  (in units of  $J/k_B$ ). The data points have been obtained from the finite-size scaling form of the order parameter in Eq. (8) and the error bars are smaller than the size of the symbol. The solid line, which is only a guide to eyes, gives the boundary separating the disordered high-temperature phase (D) from the low-temperature phase with the nonvanishing order parameter (O). Inset: The phase boundary is well described by the form  $T \approx 0.41 \ln P + 2.89$  except for the point  $P=1$ .

shown to be described by the mean-field critical exponents,  $\beta=1/2$ ,  $\bar{\nu}=2$ , and  $\alpha=0$ , irrespective of the value of  $P$ . The existence of the additional length scale corresponding to the typical distance between the ends of shortcuts is manifested by the drift of crossing points at smaller sizes. The question as to the value of  $P_c$ , below that long-range order does not emerge at any finite temperature, still remains to be answered. The phase diagram in Fig. 4 provides a very small upper bound,  $P_c \leq O(10^{-3})$ , which is likely to suggest  $P_c=0$ . In this regard, it is revealing that the small-world transition, where the characteristic path length  $\ell$  undergoes a change of behavior, occurs at  $P=0$  in the thermodynamic limit [18]. We thus suspect that the small-world transition and the order-disorder transition are intimately related, detailed investigation of which is left for further study.

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